GENERAL THEORY OF CURVES ON RULED SURFACES*

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§ 1. *Relation between the differential equations of the surface and of the curves situated upon it.*

Let a ruled surface be given by means of the system of differential equations

(1)
$$y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0,$$
$$z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0,$$

so that the curves C_y and C_z will be two curves upon it, the lines joining corresponding points of these two curves being generators of the surface. We shall eliminate once z and once y, so as to obtain the linear differential equations of the fourth order which each of these functions must satisfy.

We have from (1), by differentiation,

(2)
$$y^{(3)} = r_{11}y' + r_{12}z' + s_{11}y + s_{12}z,$$
$$z^{(3)} = r_{21}y' + r_{22}z' + s_{21}y + s_{22}z,$$

where

(3)
$$r_{11} = p_{11}^2 + p_{12}p_{21} - p_{11}' - q_{11}, \qquad s_{11} = p_{11}q_{11} + p_{12}q_{21} - q_{11}',$$
$$r_{12} = p_{12}(p_{11} + p_{22}) - p_{12}' - q_{12}, \qquad s_{12} = p_{11}q_{12} + p_{12}q_{22} - q_{12}',$$
$$r_{21} = p_{21}(p_{11} + p_{22}) - p_{21}' - q_{21}, \qquad s_{21} = p_{21}q_{11} + p_{22}q_{21} - q_{21}',$$
$$r_{22} = p_{22}^2 + p_{12}p_{21} - p_{22}' - q_{22}, \qquad s_{22} = p_{21}q_{12} + p_{22}q_{22} - q_{22}'.$$

We find, by another differentiation,

(4)
$$y^{(4)} = l_{11}y' + l_{12}z' + m_{11}y + m_{12}z,$$
$$z^{(4)} = l_{21}y' + l_{22}z' + m_{21}y + m_{22}z,$$

where

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$$
\begin{aligned}
l_{11} &= -p_{11}r_{11} - p_{21}r_{12} + r'_{11} + s_{11}, & m_{11} &= -r_{11}q_{11} - r_{12}q_{21} + s'_{11}, \\
l_{12} &= -p_{12}r_{11} - p_{22}r_{12} + r'_{12} + s_{12}, & m_{12} &= -r_{11}q_{12} - r_{12}q_{22} + s'_{12}, \\
l_{21} &= -p_{11}r_{21} - p_{21}r_{22} + r'_{21} + s_{21}, & m_{21} &= -r_{21}q_{11} - r_{22}q_{21} + s'_{21}, \\
l_{22} &= -p_{12}r_{21} - p_{22}r_{22} + r'_{22} + s_{22}, & m_{22} &= -r_{21}q_{12} - r_{22}q_{22} + s'_{22}.
\end{aligned}
$$

(5)

If we put further

(6)
$$
\Delta_1 = p_{12}s_{12} - q_{12}r_{12}, \qquad \Delta_2 = p_{21}s_{21} - q_{21}r_{21},
$$

we can find, from the above equations,

(7)
$$
\begin{aligned}
\Delta_1 z &= p_{12}y^{(3)} + r_{12}y'' + (p_{11}r_{12} - p_{12}r_{11})y' + (q_{11}r_{12} - p_{12}s_{11})y, \\
-\Delta_1 z' &= q_{12}y^{(3)} + s_{12}y'' + (p_{11}s_{12} - q_{12}r_{11})y' + (q_{11}s_{12} - q_{12}s_{11})y,
\end{aligned}
$$

and similarly,

(8)
$$
\begin{aligned}
\Delta_2 y &= p_{21}z^{(3)} + r_{21}z'' + (p_{22}r_{21} - p_{21}r_{22})z' + (q_{22}r_{21} - p_{21}s_{22})z, \\
-\Delta_2 y' &= q_{21}z^{(3)} + s_{21}z'' + (p_{22}s_{21} - q_{21}r_{22})z' + (q_{22}s_{21} - q_{21}s_{22})z.
\end{aligned}
$$

Finally, we obtain the required differential equations for y and z, viz:

(9)
$$
\begin{aligned}
\Delta_1 y^{(4)} = {}&(p_{12}m_{12} - q_{12}l_{12})y^{(3)} + (r_{12}m_{12} - s_{12}l_{12})y'' \\
&+ \big[(p_{11}r_{12} - p_{12}r_{11})m_{12} - (p_{11}s_{12} - q_{12}r_{11})l_{12} + \Delta_1 l_{11}\big]y' \\
&+ \big[(q_{11}r_{12} - p_{12}s_{11})m_{12} - (q_{11}s_{12} - q_{12}s_{11})l_{12} + \Delta_1 m_{11}\big]y,
\end{aligned}
$$

and

(10)
$$
\begin{aligned}
\Delta_2 z^{(4)} = {}&(p_{21}m_{21} - q_{21}l_{21})z^{(3)} + (r_{21}m_{21} - s_{21}l_{21})z'' \\
&+ \big[(p_{22}r_{21} - p_{21}r_{22})m_{21} - (p_{22}s_{21} - q_{21}r_{22})l_{21} + \Delta_2 l_{22}\big]z' \\
&+ \big[(q_{22}r_{21} - p_{21}s_{22})m_{21} - (q_{22}s_{21} - q_{21}s_{22})l_{21} + \Delta_2 m_{22}\big]z.
\end{aligned}
$$

These equations are capable of a vast number of applications. Any question, in fact, in regard to the existence of curves of a specified character on a ruled surface must make use of them.

We notice that the conditions $\Delta_1 = 0$ or $\Delta_2 = 0$ will be necessary and sufficient to make C_y or C_z plane curves; the differential equations (of the third order) of these plane curves are found by putting $\Delta_1 = 0$ or $\Delta_2 = 0$ in (7) or (8) respectively. We will merely indicate a few other applications of these formulas. Let us write (9), more briefly,

[(9′)]
$$
y^{(4)} + 4p_1 y^{(3)} + 6p_2 y'' + 4p_3 y' + p_4 y = 0.
$$

It is easy to write down the conditions that the integral curve of (9′) shall belong to a linear complex, or that it shall be a twisted cubic. In one case its invariant of weight 3, and in the other both of its invariants, must vanish. But

these conditions, which we now find expressed in terms of the coefficients of (1), become conditions for a particular kind of ruled surface, which contains such curves. One can impose other conditions, for example, that these curves shall be flecnode curves or asymptotic curves on the surface, and then proceed to study the particular class of surface characterized.

It is not our intention to follow up any of these *special* problems, interesting as they are. We shall, however, apply our equations for the purpose of answering some questions of a fundamental nature in the *general* theory of ruled surfaces.

§ 2. *On ruled surfaces, one of the branches of whose flecnode curve is given.*

The flecnode curve is so important in the general theory of ruled surfaces, that it seems essential to investigate to what extent it may be arbitrarily assigned.

If one of the sheets of the flecnode surface, F', of S is given, there remain only two possibilities for S, namely, one or the other of the two sheets of the flecnode surface of F'. But let us suppose that we merely know that a certain curve C is one of the branches of the flecnode curve on S. Then there are two questions to answer. Can this curve be chosen arbitrarily? And how far does it determine the surface S?

Let the curve C be given by means of its differential equation

(11) $$\frac{d^4\bar{y}}{d\bar{x}^4} + 4\bar{p}_1\frac{d^3\bar{y}}{d\bar{x}^3} + 6\bar{p}_2\frac{d^2\bar{y}}{d\bar{x}^2} + 4\bar{p}_3\frac{d\bar{y}}{d\bar{x}} + \bar{p}_4\bar{y} = 0,$$

where $\bar{p}_1, \cdots, \bar{p}_4$ are given functions of $\bar{x}$. In the system of differential equations (1) defining our surface S, we must regard the coefficients p_{ik} and q_{ik} as unknown functions. We may, however, assume without exception that $u_{12} = 0$, so that C_y is one of the branches of the flecnode curve on S, that $p_{21} = 0$, so that C_z is an asymptotic curve on S, and that $p_{11} = p_{22} = 0$. Under these assumptions we form the differential equation (9) of the curve C_y. Since C_y is to be identical with C it must be possible to convert equation (9) into (11) by a transformation of the form

(12) $$y = \phi(x)\bar{y}, \qquad \bar{x} = f(x).$$

The functions ϕ and f are not independent however. For, while the equations $u_{12} = 0$ and $p_{21} = 0$ are not disturbed by any transformation of this form, the conditions $p_{11} = p_{22} = 0$ are. In fact a transformation of the form (12) converts (1) into another system of the same form whose corresponding coefficients $\bar{p}_{11}$ and $\bar{p}_{22}$ will be

$$\bar{p}_{11} = \frac{p_{11} + 2\dfrac{\phi'}{\phi} + \dfrac{f''}{f'}}{f'}, \qquad \bar{p}_{22} = \frac{p_{22} + 2\dfrac{\phi'}{\phi} + \dfrac{f'}{f'}}{f'}.$$

In order, therefore, that after this transformation $\bar{p}_{11}$ and $\bar{p}_{22}$ may again vanish, we must have

(13)
$$\phi = \frac{C}{\sqrt{f''}},$$

where C is an arbitrary constant, which may be put equal to unity.

If then we apply the transformation (12) to (11), we shall get an equation

$$\frac{d^4y}{dx^4} + 4p_1 \frac{d^3y}{dx^3} + 6p_2 \frac{d^2y}{dx^2} + 4p_3 \frac{dy}{dx} + p_4 y = 0,$$

which we must identify with (9). Equating coefficients gives us a system of four equations with five unknown functions of x, viz.: f, p_{12}, q_{11}, q_{21}, q_{22}.

We find, therefore, the following theorem: *An arbitrary space curve being given, it can be considered as one branch of the flecnode curve of an infinity of ruled surfaces, into whose general expression there enters an arbitrary function.* One may, therefore, impose another condition and still obtain an infinity of ruled surfaces.

The most general curve C_z which is capable of being the second branch of the flecnode curve on a ruled surface for which C_y is the first branch, involves, therefore, in its expression one arbitrary function. It cannot, therefore, be an arbitrary curve, as that would involve three arbitrary functions.

Therefore, *two curves taken at random cannot be connected point to point in such a way as to constitute the complete flecnode curve upon the ruled surface thus generated.*

We can also prove our theorem by purely synthetic considerations. Let us take points P_1, P_2, P_3, P_4, $\cdots$ on an arbitrary curve, corresponding for example to equal increments Δx of the parameter. Through P_1, P_2, P_3 draw three arbitrary lines g_1, g_2, g_3. We can draw a line f_1 through P_1 intersecting g_2 and g_3, say in Q_2 and Q_3. Take an arbitrary point Q_4 on f_1, and join it to P_4 by a line g_4. Then f_1 intersects g_1, g_2, g_3, g_4. Through P_2 we draw a line f_2 intersecting g_3 and g_4 in points Q'_3, Q'_4 and of course g_2 in $Q'_2 = P_2$. Take an arbitrary point Q'_5 on f_2 and join it to P_5 by a line g_5. Continue this process. Clearly, we shall get two assemblages of lines g_1, g_2, $\cdots$ and $f_1, f_2, \cdots$, which when P_1, P_2, $\cdots$ are taken closer and closer together, approach as a limit two ruled surfaces having the given curve as flecnode curve, and which are flecnode surfaces of each other. The first three lines g_1, g_2, g_3 are arbitrary, and thus give rise to six constants of integration. Further, the double ratios (P_1, Q_2, Q_3, Q_4), (Q'_2, Q'_3, Q'_4, Q'_5), etc., may be chosen arbitrarily, which brings into evidence the arbitrary function involved in the construction of these surfaces.

The construction which has just been described becomes indeterminate if the given curve C is a straight line. For then Q_4 coincides with P_4, etc. In fact,

the most general ruled surface with a given straight line directrix depends on two arbitrary functions.

If the given curve C is to be at the same time the second branch of the flecnode curve, i. e., if both of the branches of the flecnode curve of S coincide with C, g_4 must be tangent to the hyperboloid determined by g_1, g_2, g_3; g_5 must be tangent to the hyperboloid determined by g_2, g_3, g_4; etc. This condition, therefore, clearly fixes the double ratios $(Q_1 Q_2 Q_3 Q_4)$, etc., i. e., the arbitrary function. Therefore this problem has in general ∞^6 solutions.

Let us assume that C_y is not a straight line. Let us call the developable surface formed by the tangents of C_y its *primary* developable. There exists another important developable surface containing C_y, which we shall speak of as its *secondary* developable, as indicated in the following theorem.

1. *If at every point of the flecnode curve of S there be drawn the generator of the surface, the flecnode tangent, the tangent of the flecnode curve, and finally the line which is the harmonic conjugate of the latter with respect to the other two, the locus of these last lines is a developable surface, the secondary developable of the flecnode curve.*

2. *We can find a single infinity of ruled surfaces, each having one branch of its flecnode curve in common with that of S. This family of ∞^1 surfaces can be described as an involution, of which any surface of the family and its flecnode surface form a pair. The primary and secondary developables of the branch of the flecnode surface considered, are the double surfaces of this involution. In fact, the generators of these surfaces, at every point of their common flecnode curve, form an involution in the usual sense.*

We proceed to prove these theorems. Since C_y is a branch of the flecnode curve, we may assume $u_{12} = p_{11} = p_{22} = 0$. System (1) assumes the form

$$(14) \quad y'' + p_{12}z' + q_{11}y + \tfrac{1}{2}p'_{12}z = 0, \qquad z'' + p_{21}y' + q_{21}y + q_{22}z = 0.$$

The flecnode tangent at P_y is the line joining P_y to P_ρ, where

$$\rho = 2y' + p_{12}z,$$

while the tangent of the flecnode curve joins P_y to $P_{y'}$. In the plane pencil formed by these lines, the harmonic conjugate of $P_y P_{y'}$ with respect to $P_y P_z$ and $P_y P_\rho$ will be the line $P_y P_\tau$, where

$$\tau = y' + p_{12}z.$$

But from the first equation of (14) we find

$$(15) \qquad\qquad \tau' + q_{11}y - \tfrac{1}{2}\frac{p'_{12}}{p_{12}}(\tau - y') = 0,$$

i. e., $P_y P_\tau$ generates a *developable* surface as asserted in the first theorem.

Put

(16)
$$e = \tau + ky' = (1 + k)y' + p_{12}z,$$
$$f = \tau - ky' = (1 - k)y' + p_{12}z,$$

where k is a constant. Clearly the lines $P_y P_e$ and $P_y P_f$ form a pair of the involution whose double lines are $P_y P_{y'}$ and $P_y P_\tau$.

One finds that y and e satisfy the following system of differential equations:

(17)
$$y'' + P_{11}y' + P_{12}e' + Q_{11}y + Q_{12}e = 0,$$
$$e'' + P_{21}y' + P_{22}e' + Q_{21}y + Q_{22}e = 0,$$

where

$$P_{11} = -\frac{1+k}{2k}\frac{p'_{12}}{p_{12}}, \quad P_{12} = -\frac{1}{k}, \quad Q_{11} = -\frac{q_{11}}{k}, \quad Q_{12} = \frac{1}{2k}\frac{p'_{12}}{p_{12}},$$

$$P_{21} = (1+k)q_{11} + k(1+k)q_{22} - kp_{12}p_{21} + \frac{(1-3k)(1-k^2)}{4k}\left(\frac{p'_{12}}{p_{12}}\right)^2 + \frac{1-k^2}{2}\frac{p''_{12}}{p_{12}},$$

(18) $P_{22} = \dfrac{1-3k}{2k}\dfrac{p'_{12}}{p_{12}},$

$$Q_{21} = (1+k)q'_{11} - kp_{12}q_{21} + \frac{(1-3k)(1+k)}{2k}\frac{p'_{12}}{p_{12}}q_{11},$$

$$Q_{22} = -\left[kq_{22} + \frac{(1-3k)(1-k)}{4k}\left(\frac{p'_{12}}{p_{12}}\right)^2 + \frac{1-k}{2}\frac{p''_{12}}{p_{12}}\right].$$

We find

$$U_{12} = 2P'_{12} - 4Q_{12} + P_{12}(P_{11} + P_{22}) = 0,$$

i. e., the curve C_y is flecnode curve on the ruled surface S_k generated by $P_y P_e$. The flecnode surface of S_k is obtained by joining P_y to the point

$$2y' + P_{11}y + P_{12}e = -\frac{1}{k}f - \frac{1+k}{2k}\frac{p'_{12}}{p_{12}}y,$$

a point on the line $P_y P_f$. We see therefore that the ruled surfaces S_k and and S_{-k} are flecnode surfaces of each other. We have now proved our second theorem, and we may speak of an involution of ruled surfaces having one branch of their flecnode curve in common. The double surfaces of the involution are developables, while the members of each pair of the involution are flecnode surfaces of each other.

We have seen that $P_y P_\tau$ generates a developable. If

$$g = \alpha y + \beta \tau$$

represents its edge of regression, it must be possible to represent g' in the form

$$g' = \gamma y + \delta \tau,$$

since the line $P_y P_\tau$ must then be tangent to the curve C_g.

We find, by differentiation, making use of (15),

$$g' = \alpha y' + \beta \left(\tfrac{1}{2} \frac{p'_{12}}{p_{12}} \tau - q_{11} y - \tfrac{1}{2} \frac{p'_{12}}{p_{12}} y' \right) + \alpha' y + \beta' \tau,$$

so that g' will be of the required form if, and only if,

$$\alpha : \beta = p'_{12} : 2p_{12}.$$

Therefore

$$g = p'_{12} y + 2p_{12} \tau.$$

If we express τ in terms of y, z and ρ, we shall find

(19a) $$g = p_{12}\rho + p'_{12} y + p_{12}^2 z$$

as the expression for the edge of regression of the secondary developable of the branch C_y of the flecnode curve. Similarly, if $\theta_4 \neq 0$,

(19b) $$h = p_{21}\sigma + p'_{21} z + p_{21}^2 y$$

will represent the cuspidal edge of the secondary developable of the branch C_z of the flecnode curve, assuming, of course, $u_{21} = 0$.

One easily finds

(20) $$-\tfrac{3}{2} p'_{12} g + p_{12} g' = \lambda y, \qquad -\tfrac{3}{2} p'_{21} h + p_{21} h' = \mu z,$$

where

(21)
$$\lambda = p_{12} p''_{12} + \tfrac{1}{2} p_{12}^2 u_{11} - p_{12}^3 p_{21} - \tfrac{3}{2} (p'_{12})^2,$$

$$\mu = p_{21} p''_{21} + \tfrac{1}{2} p_{21}^2 u_{22} - p_{21}^3 p_{12} - \tfrac{3}{2} (p'_{21})^2.$$

The system of differential equations of which g and h are the solutions has the coefficients

(22)
$$P_{11} = -\frac{1}{\lambda} \left[\lambda' + \tfrac{3}{4} p_{12}^2 p'_{12} p_{21} \right], \qquad P_{12} = \frac{p_{21}\lambda}{\mu}, \qquad Q_{12} = -\frac{3 p'_{21} \lambda}{2\mu},$$

$$Q_{11} = -\frac{1}{\lambda p_{12}} \left[\lambda(2p''_{12} + \tfrac{1}{4} p_{12} u_{11} - \tfrac{1}{2} p_{12}^2 p_{21}) - \tfrac{3}{2} p'_{12}(\lambda' + \tfrac{3}{4} p_{12}^2 p'_{12} p_{21}) \right],$$

while P_{21}, P_{22}, Q_{22}, Q_{21} are obtained from these same equations by permuting the indices 1 and 2, and consequently also the letters λ and μ.

We see that we thus obtain, corresponding uniquely to any ruled surface whose flecnode curve intersects every generator in two distinct points, another ruled surface which is generated by the lines joining corresponding points of the edges of regression of the secondary developables of the two branches of the flecnode curve.

Equations (20) show that one of the secondary developables of C_y and C_z degenerates into a cone if λ or μ vanishes. In that case our new ruled surface also becomes a cone. If both of the secondary developables are cones, this ruled

surface degenerates into the straight line joining their vertices. If $\lambda = \Delta_1 = 0$, the secondary developable of C_y is a plane pencil.

Equations (22) show that this new ruled surface cannot be developable except if λ or μ is zero, i. e., unless it is a cone. For the possibility $p_{12} = 0$ or $p_{21} = 0$ is to be excluded, since we should then have a ruled surface S with a straight line directrix.

§ 3. *On ruled surfaces one of the branches of whose complex curve is given.*

There exists an infinity of ruled surfaces each of which contains an arbitrarily given curve as one branch of its complex curve. Into the general analytical expression of these surfaces there enters an arbitrary function. The analytical proof of this statement is precisely similar to that of the corresponding theorem of § 2. We shall give at once a geometrical construction for these surfaces.

Let us consider five straight lines $g_1, \cdots, g_5$. Let f'_1, f''_1 be the two transversals of $g_1, \cdots, g_4$, and f'_2, f''_2 those of $g_2, \cdots, g_5$. Clearly $g_1, \cdots, g_5$ determine a linear complex, with respect to which f'_1, f''_1 and f'_2, f''_2 are two pairs of reciprocal polars. Take a point P on g_1. The plane, which corresponds to it in the linear complex, passes through g_1 and the line h, which passes through P and intersects both f'_2 and f''_2. If $g_2, \cdots, g_5$ are made to approach each other, we shall have in the limit five consecutive generators of a ruled surface and its osculating linear complex. The plane tangent to this ruled surface at P is the limit of the plane containing g_1 and the line through P which intersect g_2 and g_3, i. e., the asymptotic tangent of the surface at P. If P is a point on the complex curve, h_1 must be in the plane tangent to the ruled surface at P.

Now let an arbitrary curve be given, and let us choose points upon it, $P_1, P_2, P_3, \cdots$, according to any law. Through $P_1, \cdots, P_4$ draw four arbitrary lines $g_1, \cdots, g_4$. Through P_1 draw h_1, any line which intersects g_2. Let Q be this point of intersection. The line g_5, through P_5, is to be constructed in such a way that the two transversals of $g_2, \cdots, g_5$ shall both meet h_1. Now these transversals must be generators of the second set on the hyperboloid determined by g_2, g_3, g_4. They must, therefore, be those two generators of the second set, f'_2 and f''_2, which pass through the two points in which h_1 intersects the hyperboloid. One of these points is Q. There exists just one line through P_5 intersecting both f'_2 and f''_2. It is the line g_5. In the same way, starting with $g_2, \cdots, g_5$, we can construct g_6, etc. Finally we pass to the limit. There enters an arbitrary function, fixing the position of the successive lines $h_1, h_2, \cdots$ in the planes in which they must lie.

That a corresponding theorem is true for asymptotic curves, is obvious.

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